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A Three-Dimensional Dynamic Analysis of a Towed System

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The purpose of this investigation is to study the three-dimensional motion of a cable-body towing system. By assuming that the towline is continuous, completely flexible, and inextensible, three first-order partial differential equations of motion are developed in terms of velocity components normal and tangential to the towline. In addition to these, three kinematical first-order partial differential equations are generated to describe the position of the towline in space. The boundary conditions for the system are the motion of the towing ship and the forces exerted by the towed body. The dynamic motion of the towed system is obtained by applying the method of characteristics to the equations of motion. The characteristic equations are then numerically integrated on a computer. A transfer function, which is defined as the ratio of the resultant body amplitude of motion to the amplitude of the ship motion, is developed for various towing speeds and towline lengths. The transfer function decreases as the towline length and the towing speed increase for the examples given in this paper. Also, it is shown that, if the towline is not straight, then longitudinal disturbances produce transverse disturbances and vice versa.

Nomenclature

A_B = plan area of the towed body
A_C = cross-sectional area of towline cable
C = towline steady-state velocity with respect to the water
C_D = towline fairing drag coefficient
c = towline fairing chord
D_N = normal towline drag force per foot
E = Young's modulus for the towline
F = normal towline hydrodynamic force per foot $[(H^2 + I^2)^{1/2} = F = (D_N{}^2 + L^2)^{1/2}]$
F_B = hydrodynamic body force
$F_\alpha, F_\beta, G_\alpha, G_\beta$, $H_\alpha, H_\beta, L_\alpha, L_\beta$ = defined by Eqs. (66-74)
I, G, H = hydrodynamic towline forces in the x, y, z direction, respectively
L = towline lift force per foot
L_C = total length of towline
M_B = mass of body
R = $(U^2 + W^2)^{1/2}$
R_e = Reynolds number
R_d = $\rho/2cC_DR^2$
s = arc length along towline
t = time
U, V, W = velocity components in the x, y, z direction, respectively
u, v, w = velocity components in the X, Y, Z direction, respectively
VOW = $V_0 \cos\theta_0 - W_0 \sin\theta_0$
W_B = weight of body in water
W_t = weight of towline per foot in water
X, Y, Z = spatial coordinates of the towline
x, y, z = coordinates aligned with the towline
δ = boundary-layer thickness
δK = time increment for numerical solution
δh = arc length increment for numerical solution
ϕ, θ = directional angles of the towline
ρ = density of fluid
μ = mass of towline per foot
μ_f = viscosity of fluid

Subscripts

d = dynamic properties
L = boundary condition at the body
T = directed along the towline
u = condition at the surface
0 = steady-state condition
1 = small perturbation about a steady-state value
α = directed along the α characteristic line

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β = directed along the β characteristic line
x, y, z = components in the x, y, z direction, respectively

1. Introduction

THE purpose of this investigation is to study the three-dimensional dynamic motion of a cable-body towing system. Such devices have aerodynamic application in glider towing, air-to-air refueling, and airplane-towed sonar devices for submarine detection. At this time, there is a great deal of interest in ocean exploration. Buoys, diving sources, and midget submarines have been employed to study the ocean. Presently, towed bodies have had limited application in this work. However, some military use has been made of cable-towed systems for mine sweeping and sonar detection. In order to exploit the advantages of towed systems for oceanographic exploration, the problem of accurately defining the system's configuration and determining design criteria to insure the stability of the towed system must be solved.

Many investigators have studied the problem of towed bodies. The earliest investigators, such as O'Hara,[1] were concerned with airborne, towed vehicles for glider application during the second World War. The first information on underwater, towed bodies was contained in Pode's[2] work on the two-dimensional equilibrium configuration of a cable immersed in a steady uniform stream. The hydrodynamic forces acting on the towline were divided into a normal and a tangential component. The normal component was taken to be proportional to the square of the sine of the towline angle, whereas a constant was employed for the tangential component. His report contained numerical tables of steady-state towline configurations.

This work formed the basis of most towed-body calculations until 1957. At that time Whicker[3] developed a solution for the two-dimensional equilibrium configuration of the towline with improved hydrodynamic loading functions. He also indicated a method of solution for the two-dimensional dynamic motion of the towed system.

Walton[4] developed a numerical method for calculating the dynamic tension in a mooring cable for ships taking part in atomic bomb experiments. He assumed that the mooring cable was made up of many discrete masses. The tangential hydrodynamic towline force was neglected, and Pode's form of the normal force component was applied. Numerical results of tension were obtained for various sea states.

One of the first attempts to formulate the three-dimensional motion of a towed system was made by Strandhagen.[5] In his paper Pode's hydrodynamic loading functions were used. The towline was approximated by rigid links joined together by universal joints. A complicated solution for the acceleration of the towed body and cable elements was indicated but not numerically evaluated.

An extension of Strandhagen's formulation was made by The Boeing Company.[6] In this report, the universal joints were replaced by springs and dashpots to allow moments to be transported from link to link. The motion of the system was obtained with the use of an analog computer.

In the present study, assuming that the towline is continuous, inextensible, and flexible, three first-order partial differential equations of motion are developed. In addition, three kinematical first-order partial differential equations are generated to describe the position of the towline in space. The boundary conditions for the system are the motion of the towing ship and the forces acting on the towed body. The dynamic motion of the towed system is obtained by applying the method of characteristics to the equations of motion and the kinematic equations. Then the characteristic equations are numerically integrated on a computer.

The numerical examples presented in this paper are based on a system towed by the U.S. Navy. Physically, the towed system is illustrated in Fig. 1, and it consists of a ship, a towline, and a submerged body. The towline is made up of a twisted wire-rope cable and segmented fairing sections as shown in Fig. 2.

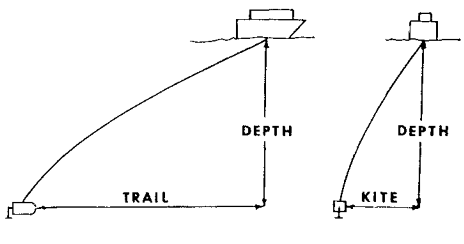

Fig. 1 Towed system.

2. Equations of Motion

2.1 Mathematical Formulation

The towline is allowed to have a three-dimensional configuration in space. Its orientation is described by the angles ϕ and θ, which are functions of time t and arc length s. In order to facilitate a solution of the equations of motion, the space reference system X, Y, Z is transformed to a coordinate system x, y, z. As indicated in Fig. 3, the y coordinate is directed along the towline, whereas the x and z coordinates are perpendicular to the towline. It should be noted that, as θ goes to zero, the solution reduces to the two-dimensional case.

The hydrodynamic mass of the towline will be considered negligible. Since the fairing has a streamline shape, this restriction will have little effect on the motion of the system in the towing plane. However, the dynamic motion of the system out of the towing plane may be slightly overestimated. This assumption is made because there is little data available on the hydrodynamic mass of the fairing, and their inclusion would greatly complicate the numerical solution for the dynamic motion of the system.

2.2 Derivation of Equations

A free-body diagram of an infinitesimal length ds of the towline is shown in Fig. 3a. Applying Newton's second law of motion to the element along the space axis X, Y, and Z, one obtains

$$\mu \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{T}{(1+e)}\frac{\partial X}{\partial s}\right) + F_x \tag{1}$$

$$\mu \frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{T}{(1+e)}\frac{\partial Y}{\partial s}\right) + F_y \tag{2}$$

$$\mu \frac{\partial^2 Z}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{T}{(1+e)}\frac{\partial Z}{\partial s}\right) + F_z \tag{3}$$

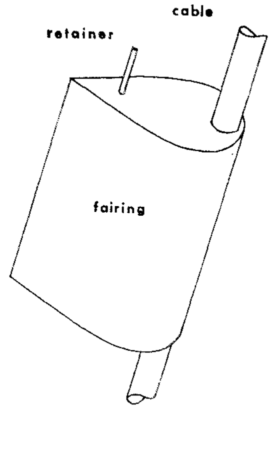

Fig. 2 Towline.

Restricting the problem to an inextensible towline and expanding the force terms, Eqs. (1–3) become

$$\mu \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial X}{\partial s}\right) + G \cos\theta \cos\phi - I \sin\phi + H \sin\theta \cos\phi \quad (4)$$

$$\mu \frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial Y}{\partial s}\right) + G \cos\theta \sin\phi + I \cos\phi + H \sin\phi \sin\theta - W_t \quad (5)$$

$$\mu \frac{\partial^2 Z}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial Z}{\partial s}\right) - H \cos\theta + G \sin\theta \quad (6)$$

As illustrated in Fig. 3b, the directional derivatives of the towline are

$$\frac{\partial X}{\partial s} = \cos\theta \cos\phi \qquad \frac{\partial Y}{\partial s} = \cos\theta \sin\phi \qquad \frac{\partial Z}{\partial s} = \sin\theta \quad (7)$$

Also the velocity components in the space coordinate system for the center of gravity of the towline element are defined as

$$u = \partial X/\partial t \qquad v = \partial Y/\partial t \qquad w = \partial Z/\partial t \quad (8)$$

Substituting Eqs. (8) and (7) into (4–6) yields

$$\mu \frac{\partial u}{\partial t} = \frac{\partial}{\partial s}(T \cos\theta \cos\phi) + G \cos\theta \cos\phi - I \sin\phi + H \sin\theta \cos\phi \quad (9)$$

$$\mu \frac{\partial v}{\partial t} = \frac{\partial}{\partial s}(T \cos\theta \sin\phi) + G \cos\theta \sin\phi + I \cos\phi + H \sin\theta \sin\phi - W_t \quad (10)$$

$$\mu \frac{\partial w}{\partial t} = \frac{\partial}{\partial s}(T \sin\theta) + G \sin\theta - H \cos\theta \quad (11)$$

The solution to the preceding equations of motion is simplified if a coordinate transformation is made from the space reference X, Y, Z to one aligned with the towline element x, y, z as indicated in Fig. 3b. This change of coordinates can be accomplished by employing the following transformation matrix [ ] indicate square matrix, { } indicate column matrix):

$$[A] = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi \cos\theta & \cos\theta \sin\phi & \sin\theta \\ -\cos\phi \sin\theta & -\sin\theta \sin\phi & \cos\theta \end{bmatrix}$$

If Eqs. (9–11) are represented by a column matrix such as

$$\begin{Bmatrix} (9) \\ (10) \\ (11) \end{Bmatrix}$$

the transformed equations are written as follows:

$$\begin{Bmatrix} (12) \\ (13) \\ (14) \end{Bmatrix} = [A] \begin{Bmatrix} (9) \\ (10) \\ (11) \end{Bmatrix}$$

After performing the matrix multiplication and simplifying the expressions, the equations of motion in the x, y, z coordinate system become

$$\mu\left(\frac{\partial v}{\partial t} \cos\phi - \frac{\partial u}{\partial t} \sin\phi\right) = T \cos\theta \frac{\partial \phi}{\partial s} + I - W_t \cos\phi \quad (12)$$

$$\mu\left[\left(\frac{\partial u}{\partial t} \cos\phi + \frac{\partial v}{\partial t} \sin\phi\right) \cos\theta + \frac{\partial w}{\partial t} \sin\theta\right] = \frac{\partial T}{\partial S} + G - W_t \sin\phi \cos\theta \quad (13)$$

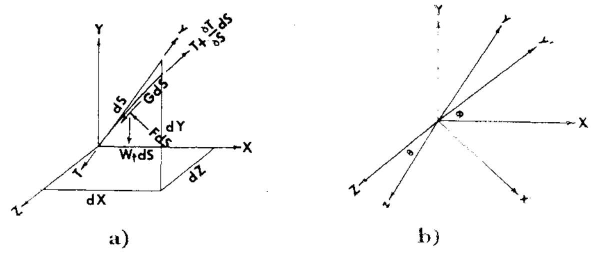

Fig. 3 a) Towline forces. b) Coordinate transformation.

$$\mu\left[\left(\frac{\partial u}{\partial t} \cos\phi + \frac{\partial v}{\partial t} \sin\phi\right) \sin\theta - \frac{\partial w}{\partial t} \cos\theta\right] = -T \frac{\partial \theta}{\partial s} + H - W_t \sin\phi \sin\theta \quad (14)$$

The velocity components u, v, w are transformed to the velocity components U, V, and W in the x, y, z directions as follows:

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = [A] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Hence, the transformed velocities are

$$U = u \sin\phi - v \cos\phi \quad (15)$$

$$V = u \cos\phi \cos\theta + v \sin\phi \cos\theta + w \sin\theta \quad (16)$$

$$W = -u \sin\theta \cos\phi - v \sin\theta \sin\phi + w \cos\theta \quad (17)$$

With the use of the preceding relationships, the left side of Eqs. (12–14) can be converted to functions of U, V, W, θ, and ϕ and their time derivatives. This is accomplished by developing the following identities:

$$\frac{\partial v}{\partial t} \cos\phi - \frac{\partial u}{\partial t} \sin\phi = -\frac{\partial U}{\partial t} + (V \cos\theta - W \sin\theta) \frac{\partial \phi}{\partial t} \quad (18)$$

$$\left(\frac{\partial u}{\partial t} \cos\phi + \frac{\partial v}{\partial t} \sin\phi\right) \cos\theta + \frac{\partial w}{\partial t} \sin\theta = \frac{\partial V}{\partial t} - W \frac{\partial \theta}{\partial t} + U \cos\theta \frac{\partial \phi}{\partial t} \quad (19)$$

$$\left(\frac{\partial u}{\partial t} \cos\phi + \frac{\partial v}{\partial t} \sin\phi\right) \sin\theta - \frac{\partial w}{\partial t} \cos\theta = -\frac{\partial W}{\partial t} - V \frac{\partial \theta}{\partial t} + U \sin\theta \frac{\partial \phi}{\partial t} \quad (20)$$

Therefore, the equations of motion for the towline element written in the x, y, z coordinate system are

$$\mu\left[-\frac{\partial U}{\partial t} + (V \cos\theta - W \sin\theta) \frac{\partial \phi}{\partial t}\right] = T \cos\theta \frac{\partial \phi}{\partial s} + I - W_t \cos\phi \quad (21)$$

$$\mu\left[\frac{\partial V}{\partial t} - W \frac{\partial \theta}{\partial t} + U \cos\theta \frac{\partial \phi}{\partial t}\right] = \frac{\partial T}{\partial s} + G - W_t \sin\phi \cos\theta \quad (22)$$

$$\mu\left[-\frac{\partial W}{\partial t} - V \frac{\partial \theta}{\partial t} + U \sin\theta \frac{\partial \phi}{\partial t}\right] = -T \frac{\partial \theta}{\partial s} + H - W_t \sin\phi \sin\theta \quad (23)$$

Three additional equations of a kinematical nature, which are needed to describe the towline's position in space, can be developed by taking the partial derivative with respect to s of the transformed velocities. With the aid of the directional derivatives, the velocities u, v, and w are eliminated from the resulting equations. The derivation of the kinematical equations will be illustrated for U.

Taking the partial derivative of Eq. (15) with respect to s, one obtains

$$\frac{\partial U}{\partial s} = \frac{\partial u}{\partial s} \sin\phi - \frac{\partial u}{\partial d} \cos\phi + u \cos\phi \frac{\partial \phi}{\partial s} + u \sin\phi \frac{\partial \phi}{\partial s} \quad (24)$$

The quantity $\partial u/\partial s$ can be written as

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\partial X}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial s}\right) = \frac{\partial}{\partial t}(\cos\theta \cos\phi) \quad (25)$$

Using similar definition for $\partial v/\partial s$, one can obtain the following identity:

$$\frac{\partial u}{\partial s} \sin\phi - \frac{\partial v}{\partial s} \cos\phi = -\cos\theta \frac{\partial \phi}{\partial t} \quad (26)$$

The last two terms on the right side of (21) can be related as

$$u \cos\phi \frac{\partial \phi}{\partial s} + v \sin\phi \frac{\partial \phi}{\partial s} = (V \cos\theta - W \sin\theta) \frac{\partial \phi}{\partial s} \quad (27)$$

Substituting (26) and (27) into (24), one has

$$\frac{\partial U}{\partial s} = -\frac{\partial \phi}{\partial t} \cos\theta + (V \cos\theta - W \sin\theta) \frac{\partial \phi}{\partial s} \quad (28)$$

In a like manner the following are derived:

$$\partial V/\partial s = W(\partial\theta/\partial s) - U \cos\theta(\partial\phi/\partial s) \quad (29)$$

$$\partial W/\partial s = (\partial\theta/\partial t) - V(\partial\theta/\partial s) + U \sin\theta(\partial\phi/\partial s) \quad (30)$$

Equations (21–23 and 28–30) represent the six first-order partial differential equations that describe the motion of the towline.

2.3 Boundary Conditions

The towline at its point of attachment to the towing ship must have the same motion as the ship. This constraint provides the upper boundary condition for the towline. The ship motion at the tow point can be given by

$$u = C + u^* \quad (31)$$

$$v = v^* \quad (32)$$

$$w = w^* \quad (33)$$

where u^*, v^*, and w^* are the velocity variations about the steady towing speed C. These time-dependent velocity variations are due to the wave motion of the sea, which causes the towing vessel to pitch, roll, and heave. For convenience, these motions are assumed to have a sinusoidal form in all of the numerical examples. However, any ship motion can be easily introduced into the boundary conditions.

In order to make use of these boundary conditions in the numerical solution, a transformation of Eqs. (31–33) from the spatial coordinate system to the system aligned with the towline is made. The upper boundary conditions become

$$U_u = (C + u^*) \sin\phi - v^* \cos\phi \quad (34)$$

$$V_u = (C + u^*) \cos\theta \cos\phi + v^* \cos\theta \sin\phi + w^* \sin\theta \quad (35)$$

$$W_u = -(C + u^*) \sin\theta \cos\phi - v^* \sin\theta \sin\phi + w^* \cos\theta \quad (36)$$

At the lower end, the towline must have the same motion as the body at its point of attachment. For convenience of computation, the tow point is considered to coincide with the center of mass of the body. This assumption eliminates the effect of the rotational modes of the body (pitch, roll, and yaw) on the motion of the towline. The transformation of the velocity components from the body coordinate system to the towline coordinate system is accomplished as follows:

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix}_{\text{towline}} = [A] \quad [E] \begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix}_{\text{body}}$$

The equations that form the lower boundary conditions represent Newton's second law of motion for the body forces and moments referred to a coordinate system fixed to the body at its center of mass. $[E]$ is the matrix that transforms the body coordinate system to the space system.

The development of the hydrodynamic towline forces G, H, and I is given by the author in Ref. 9. These forces are found to be a function of U, V, W, ϕ, θ, and constants.

3. Dynamic Towline Motion

3.1 Application of the Method of Characteristics

The system of quasi-linear partial differential equations developed in Sec. 2.2 represents the motion of the towline as a function of time and position. The solution of these equations by the method of characteristics will not restrict the motion of the system to small perturbations about its equilibrium state. In this section the characteristic equations of the towline will be developed.

The method of characteristics can only be applied to hyperbolic partial differential equations. However, Eqs. (22) and (29) are of the parabolic type as evidenced by their zero characteristic roots. These zero roots are a consequence of the assumption that the towline is inextensible. Therefore, these equations will not be transformed by this method.

In the hyperbolic equations (21, 23, 28, and 30), the dependent variables appear in partial derivatives with respect to time and arc length. It would simplify matters if these equations could be replaced by equivalent expressions containing only total derivatives in a given direction in the $t - s$ plane. The lines in the $t - s$ plane having these directions are called characteristic lines. The set of characteristic equations equivalent to (21, 23, 28, and 30), and the characteristic directions are determined as follows.

As indicated in Ames,[8] the characteristic equations can be obtained by writing the hyperbolic equations in matrix form:

$$[A']\{\alpha_s\} + [B']\{\alpha_t\} + \{d\} = \{0\} \quad (37)$$

where the subscript represents the partial derivative and

$$[A'] = \begin{bmatrix} 0 & 0 & -T/\mu \cos\theta & 0 \\ 0 & 0 & 0 & T/\mu \\ 1 & 0 & -(V\cos\theta - W\sin\theta) & 0 \\ 0 & 1 & -V\sin\theta & V \end{bmatrix} \quad (38)$$

$$[B'] = \begin{bmatrix} -1 & 0 & (V\cos\theta - W\sin\theta) & 0 \\ 0 & -1 & U\sin\theta & -V \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (39)$$

$$\{\alpha\} = \begin{Bmatrix} U \\ W \\ \phi \\ \theta \end{Bmatrix} \quad (40)$$

$$\{d\} = \begin{Bmatrix} -[I - W_t \cos\phi] \\ +[-H + W_t \sin\phi \sin\theta] \\ 0 \\ 0 \end{Bmatrix} \quad (41)$$

Next, Eq. (37) will be multiplied by a linear transformation matrix $[T]$;

$$[T][A']\{\alpha_s\} + [T][B']\{\alpha_t\} + [T]\{d\} = 0 \quad (42)$$

The matrix $[T]$ is chosen such that,

$$[T][A'] = [\lambda][T][B'] \quad (43)$$

where $[\lambda]$ is a diagonal matrix. Substituting (43) into (42) and defining $[T][B'] = [A^*]$ and $[T]\{d^*\}$, one obtains

$$[\lambda][A^*]\{\alpha_s\} + [A^*]\{\alpha_t\} + \{d^*\} = 0 \quad (44)$$

The jth equation becomes

$$\sum_{i=1}^{n} a_{ji}\,(\lambda_j\alpha_s{}^i + \alpha_t{}^i) + d_j = 0 \quad (45)$$

Let (M,N) be a unit vector in the $t - s$ plane (see Fig. 4), for which

$$\lambda_j = M/N = \cot\theta' \quad (46)$$

Now one could write

$$\lambda_j\alpha_s{}^i + \alpha_t{}^i = \frac{1}{N}\,(M\alpha_s{}^i + N\alpha_t{}^i) = \frac{1}{N}\,(\alpha_s{}^i\cos\theta' + \alpha_t{}^i\sin\theta') \quad (47)$$

which, except for $1/N$, is the directional derivative in the direction of the vector defined by (M,N). As indicated in Eq. (46), the direction is a function of λ_j. Hence, one can conclude that every equation of the transformed system (44) contains derivatives only in a given characteristic direction.

The characteristic directions can be found by rearranging Eq. (43) to read

$$[T]([A'] - [\lambda][B']) = [0] \quad (48)$$

If the transformation matrix $[T]$ were the zero matrix, a trivial solution would result. Consequently, it is concluded that

$$|[A'] - [\lambda][B']| = 0 \quad (49)$$

After evaluating the above 4×4 determinant, the characteristic roots are found to be

$$\lambda_{1,2} = (T/\mu)^{1/2} \qquad \lambda_{3,4} = -(T/\mu)^{1/2} \quad (50)$$

The characteristic roots or directions are real, which indicates that Eqs. (21, 23, 28, and 30) are hyperbolic partial differential equations. The characteristic values represent the speed at which transverse motions are propagated along the towline. However, Eq. (50) shows that the characteristic roots are repeated. Physically, this means that the speed at which transverse disturbances are propagated in the $x - y$ and the $z - y$ planes are the same. This property simplifies the numerical solution of the characteristic equations that will be presented in Sec. 3.2.

In order to find the characteristic equations equivalent to (21, 23, 28, and 30), Eq. (21) multiplied by dt is subtracted from Eq. (28) multiplied by ds to yield

$$\frac{\partial U}{\partial t}\,dt - (V\cos\theta - W\sin\theta)\,\frac{\partial\phi}{\partial t}\,dt + \frac{T}{\mu}\cos\theta\,\frac{\partial\phi}{\partial s}\,dt + \frac{1}{\mu}\,(I - W_t\cos\phi)\,dt + \frac{\partial U}{\partial s}\,ds + \frac{\partial\phi}{\partial t}\cos\theta\,ds - (V\cos\theta - W\sin\theta)\,\frac{\partial\phi}{\partial s}\,ds = 0 \quad (51)$$

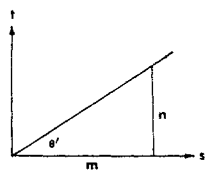

Fig. 4 Vector in the t-s plane.

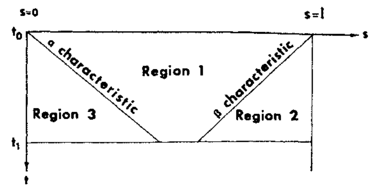

Fig. 5 Characteristic lines in the t-s plane.

Since there are two independent variables s and t, the total differential of any dependent variable, say U, is

$$dU = (\partial U/\partial s)ds + (\partial U/\partial t)dt \quad (52)$$

Hence, Eq. (51) can be simplified to yield

$$dU - (V\cos\theta - W\sin\theta)d\phi + \frac{T}{\mu}\cos\theta\,\frac{\partial\phi}{\partial s}\,dt + \frac{\partial\phi}{\partial t}\cos\theta ds + \frac{1}{\mu}\,(I - W_t\cos\phi)dt = 0 \quad (53)$$

Also, it was found in Eq. (50) that $(T/\mu)^{1/2}$ is the propagation velocity of a disturbance in the $t - s$ plane. Hence, one can write

$$ds/dt = \pm(T/\mu)^{1/2} \quad (54)$$

With the use of Eq. (54), the second and third terms of (53) can be written as

$$\cos\theta\left[\frac{T}{\mu}\left(\frac{\partial\phi}{\partial s}\right)dt + \frac{\partial\phi}{\partial t}\,ds\right] = \frac{ds}{dt}\cos\theta d\phi = \pm\left(\frac{T}{\mu}\right)^{1/2}\cos\theta d\phi \quad (55)$$

Substituting (55) into (53) yields

$$dU + d\phi\left[\pm\left(\frac{T}{\mu}\right)^{1/2}\cos\theta - (V\cos\theta - W\sin\theta)\right] + \frac{1}{\mu}\,(I - W_t\cos\phi)dt = 0 \quad (56)$$

Equation (56) represents the differential equations of the transverse motion of the towline in the $x - y$ plane. The differentiation is taken along characteristic directions. Referring to Fig. 5, the α characteristic line is associated with $(T/\mu)^{1/2}$ and represents disturbances traveling up the towline. On the other hand, the β characteristic is associated with $-(T/\mu)^{1/2}$ and represents the propagation of disturbances down the towline. Therefore, the characteristic equations become

$$\frac{dU}{d\alpha} + \frac{d\phi}{d\alpha}\left[\left(\frac{T}{\mu}\right)^{1/2}\cos\theta - (V\cos\theta - W\sin\theta)\right] + \frac{dt}{d\alpha}\,(I - W_t\cos\phi)\,\frac{1}{\mu} = 0 \quad (57)$$

$$\frac{dU}{d\beta} - \frac{d\phi}{d\beta}\left[\left(\frac{T}{\mu}\right)^{1/2}\cos\theta + (V\cos\theta - W\sin\theta)\right] + \frac{dt}{d\beta}\cdot(I - W_t\cos\phi)\,\frac{1}{\mu} = 0 \quad (58)$$

Performing the same type of mathematical manipulations as the preceding on Eqs. (23) and (30), which represent the transverse propagation of a disturbance along the towline in the $z - y$ plane, one obtains

$$\frac{dW}{d\alpha} + V\,\frac{d\theta}{d\alpha} - \left(\frac{T}{\mu}\right)^{1/2}\frac{d\theta}{d\alpha} - U\sin\theta\,\frac{d\phi}{d\alpha} - \frac{1}{\mu}\,(W_t\sin\phi\sin\theta - H)\,\frac{dt}{d\alpha} = 0 \quad (59)$$

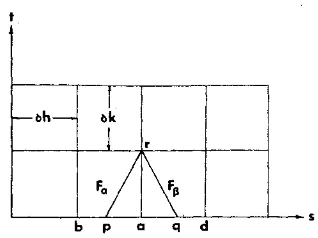

Fig. 6 Mesh for numerical computation.

$$\frac{dW}{d\beta} + V\frac{d\theta}{d\beta} + \left(\frac{T}{\mu}\right)^{1/2}\frac{d\theta}{d\beta} - U\sin\theta\frac{d\phi}{d\beta} - \frac{1}{\mu}(W_t\sin\phi\sin\theta - H)\frac{dt}{d\beta} = 0 \quad (60)$$

where α and β are the characteristic directions in the $z - y$ plane.

Since the characteristic roots are repeated, the change in α along a characteristic line in the $x - y$ plane is equal to the change in the $z - y$ plane;

$$\left.\frac{d\phi}{d\alpha}\right|_{x-y} = \left.\frac{d\phi}{d\alpha}\right|_{z-y} \qquad \left.\frac{d\phi}{d\beta}\right|_{x-y} = \left.\frac{d\phi}{d\beta}\right|_{z-y} \quad (61)$$

Equations (61) provide the basis for the numerical solution of the three-dimensional equations of motion of the towline which is presented in Sec. 3.2.

3.2 Numerical Procedure for Dynamic Solution

In order to devise systematically a numerical procedure to solve the equations of motion on a digital computer, the towline will be divided into three regions (see Fig. 5). Since region I is interior to the boundaries, the solution in this area can be advanced to the next time line t_1 using only the characteristic equations. These equations can be written in the following form:

$$dU + G_\alpha d\phi + H_\alpha dt = 0 \quad (62)$$

$$dU + G_\beta d\phi + H_\beta dt = 0 \quad (63)$$

$$dW + J_\alpha d\theta + K_\alpha d\phi + L_\alpha dt = 0 \quad (64)$$

$$dW + J_\beta d\theta + K_\beta d\phi + L_\beta dt = 0 \quad (65)$$

where α and β refer to the two characteristic directions and

$$J_\alpha = [V - (T/\mu)^{1/2}] \quad (66)$$

$$J_\beta = [V + (T/\mu)^{1/2}] \quad (67)$$

$$K_\alpha = K_\beta = -U\sin\theta \quad (68)$$

$$L_\alpha = L_\beta = 1/\mu\,(H - W_t\sin\phi\sin\theta) \quad (69)$$

$$G_\alpha = W_\alpha\sin\theta - V\cos\theta + (T/\mu)^{1/2}\cos\theta \quad (70)$$

$$G_\beta = W\sin\theta - V\cos\theta - (T/\mu)^{1/2}\cos\theta \quad (71)$$

$$H_\alpha = H_\beta = (1/\mu)(I - W_t\cos\phi) \quad (72)$$

Also the propagation speed of a disturbance traveling up or down the towline is abbreviated as

$$F_\alpha = (T/\mu)^{1/2} \quad (73)$$

$$F_\beta = -(T/\mu)^{1/2} \quad (74)$$

The numerical solution of the preceding equations is based on a scheme suggested by Courant, Isaacson, and Rees, which is presented in Ames.[8] In this method the mesh points are defined in advance (see Fig. 6), which avoids the need for a four-dimensional interpolation to advance the solution to the next mesh point. However, this numerical scheme has a first-order truncated error.

The size of the grid, i.e., the ratio of the time increment to the space increment, is dictated by two considerations. Ames shows that the stability and the convergence of the numerical technique used to integrate the characteristic equations are insured if

$$|F_\alpha|(\delta K/\delta h) < 1 \quad \text{and} \quad |F_\beta|(\delta K/\delta h) < 1$$

Secondly, the magnitude of the time interval must be sufficiently small that the characteristic lines between mesh points can be assumed straight. Through a process of trial and error, the following parameters are chosen: $\delta K = 0.005$ sec and $\delta h = 5.0$ ft.

The initial values of the towline parameters are obtained from the steady-state computer solution in Ref. 9. The first step in the numerical solution is to locate points p and q as shown in Fig. 6. Since all grid points, i.e., a, d, r, etc., are defined in advance and F_α and F_β are assumed to be straight between time lines, the points p and q are found as follows:

$$r = p = \delta KF_\alpha \quad (75)$$

$$r - q = \delta KF_\beta \quad (76)$$

The value of the dependent variables at p and q are obtained by a straight line interpolation between a and d and a and b, respectively. Next, Eqs. (62) and (63) can be written in difference notation as

$$U_r - U_q + G_\alpha(a)[\phi_r - \phi_p] + H_\alpha(a)dt = 0 \quad (77)$$

$$U_r - U_p + G_\beta(a)[\phi_r - \phi_q] + H_\beta(a)dt = 0 \quad (78)$$

Equations (77) and (78) contain two unknowns, U_r and ϕ_r, and they are evaluated as follows:

$$\phi_r = \frac{U_p - U_q + G_\beta(a)\phi_q - G_\alpha(a)\phi_p}{[G_\alpha(a) - G_\beta(a)]} \quad (79)$$

$$U_r = \{U_pG_\beta - U_qG_\alpha(a) + G_\beta(a)G_\alpha(a)[\phi_p - \phi_q] + H_\alpha(a)dt[G_\beta(a) - G_\alpha(a)]\}/[G_\beta(a) - G_\alpha(a)] \quad (80)$$

Repeating the preceding procedure for Eqs. (64) and (65),

$$W_r - W_p + J_\alpha(a)[\theta_r - \theta_p] + K_\alpha(a)[\phi_r - \phi_p] + L_\alpha(a)dt = 0 \quad (81)$$

$$W_r - W_q + J_\beta(a)[\theta_r - \theta_q] + K_\beta(a)[\phi_r - \phi_q] + L_\alpha(a)dt = 0 \quad (82)$$

Since the characteristic roots are repeated, the variable ϕ_r, found in Eq. (79), can be used in Eqs. (81) and (82). Consequently, W_r and θ_r are the only unknowns in the preceding equations, and they can be determined as shown below:

$$W_r = \{J_\beta(a)W_p - J_\alpha(a)W_q + J_\alpha(a)J_\beta(a)[\theta_p - \theta_q] + J_\beta(a)K_\alpha(a)[\phi_p - \phi_r] + J_\alpha(a)K_\beta(a)[\phi_r - \phi_q] + L_\alpha(a)dt[J_\beta(a) - J_\alpha(a)]\}/[J_\beta(a) - J_\alpha(a)] \quad (83)$$

$$\theta_r = \{\theta_pJ_\alpha(a) - \theta_qJ_\beta(a) + W_p - W_q - \phi_r[K_\alpha(a) - K_\beta(a)] + \phi_pK_\alpha(a) - \phi_qK_\beta(a)\}/[J_\alpha(a) - J_\beta(a)] \quad (84)$$

In region 2 (see Fig. 7a), the α characteristic equations are matched with the upper boundary conditions, which are described in Sec. 2.3. The upper boundary equation (34) and

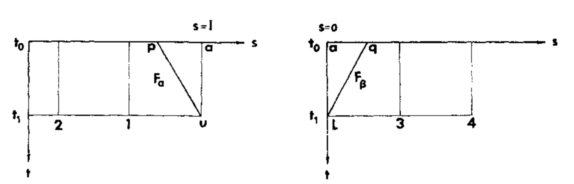

Fig. 7 a) Upper boundary. b) Lower boundary.

the characteristic equation (77) in $x - y$ plane for region 2 become

$$U_u = (C + u^*)\sin\phi_u + v^*\cos\phi_u \quad (85)$$

$$U_u - U_p + G_\alpha(a)[\phi_u + \phi_p] + H_\alpha(a)dt = 0 \quad (86)$$

Eliminating U_u from the preceding equations yields the following transcendental equation:

$$\phi_u = \{U_p + G_\alpha(a)\phi_p - H_\alpha(a)dt - [C + u^*]\sin\phi_u + v^*\cos\phi_u\}/G_\alpha(a) \quad (87)$$

An iterative procedure is used to solve for ϕ_u. This value of ϕ_u is then substituted into Eq. (51) to give U_u. The values of W_u and θ_u are obtained in a similar manner using Eqs. (36) and (81).

$$W_u = -[C + u^*]\sin\theta_u\cos\phi_u + w^*\cos\theta_w - v\sin\phi_u\sin\theta_u \quad (88)$$

$$W_u - W_p + J_\alpha(a)[\theta_u - \theta_p] + K_\alpha(a)[\phi_u - \phi_p] + L_\alpha(a)dt = 0 \quad (89)$$

where ϕ_u was evaluated in the previous calculations. Therefore, W_u and θ_u can be evaluated from the preceding equations.

Before proceeding to the lower boundary condition, V must be determined along the towline by using the upper boundary condition, Eq. (35), and the parabolic equation (29). Referring to Fig. 7a, Eq. (29) can be written in finite-difference form as

$$V_1 = V_u + \tfrac{1}{2}[U_u + U_1](\cos\theta_u + \theta_1)[\phi_u - \phi_1] - \tfrac{1}{2}[W_u + W_1][\theta_u - \theta_1] \quad (90)$$

Using this equation, V can be found at all locations along the towline except at the lower boundary.

In order to determine the values of the dependent variables at the lower boundary, nine simultaneous equations must be solved. Referring to Fig. 7, three of these equations are obtained by writing Eqs. (63, 65, and 90) in finite-difference form at the lower boundary;

$$U_L - U_q + G_\beta(a)[\phi_L - \phi_q] + H_\beta(a)dt = 0$$

$$W_L - W_q + J_\beta(a)[\theta_L - \theta_q] + K_\beta(a)[\phi_L - \phi_q] + L_\alpha(a)dt = 0$$

$$V_L - V_4 - W_3[\theta_L - \theta_4] + U_3\cos\theta_3[\phi_L - \phi_4] = 0$$

The other six equations are the three-dimensional rectilinear and rotational equations of motion of the body which are developed in Appendix B of Ref. 9. The unknowns in these equations are U_L, V_L, W_L, ϕ_L, θ_L, T_L, and the body's pitch, roll, and yaw.

Once the tension is found at L, the tension at any point along the towline can be computed by using the parabolic equation (22) in finite-difference form. To evaluate the tension at point 3 in Fig. 7b, the following approximation to (22) is made:

$$T_3 = T_L + \delta h\left\{\frac{\mu}{\delta K}[V_3 - V_d - W_3(\theta_3 - \theta_d) + U_3\cos\theta_3(\phi_3 - \phi_d)] + W_t\sin\phi_3\cos\theta_3\right\}$$

Table 1
Propagation of longitudinal and transverse motion[a]

Disturbance at surface, ft		Disturbance at body, ft	
Along towline	Transverse to towline	Along towline	Transverse to towline
10	0	9.33	2.05
0	10	2.60	4.22

[a] These results are for a 400-ft towline being towed at 20 knots.

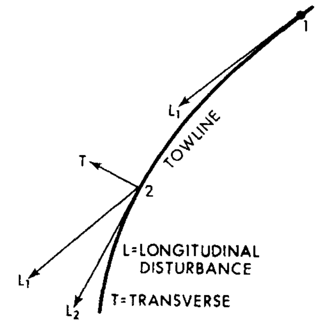

Fig. 8 Propagation of disturbances along the towline.

Then the tension can be evaluated at each mesh point going up the towline as illustrated below for T_4:

$$T_4 = T_L + 2\delta h\left\{W_t\sin\phi_3\cos\theta_3 + \frac{\mu}{\delta K}[V_3 - V_d - W_3(\theta_3 - \theta_d) + U_3\cos\theta_3(\phi_3 - \phi_d)]\right\}$$

Therefore, all the variables are determined on time line t_1. This procedure can be repeated from one time line to the next for any desired length of time. The preceding solution has been programmed on an IBM 7044, using the FORTRAN 4 language.

3.3 Numerical Results

The nonhomogeneous boundary conditions are introduced to the system as the variation of the ship's velocity about its steady-state towing speed. Since the disturbance to the system is initiated at a point in time where the velocities at the upper boundary are zero, an initial discontinuity in the acceleration of the system is experienced. The computer results indicate that the acceleration of the system goes from zero to its proper value in just a few steps or time lines. Therefore, the initial discontinuity in acceleration has very little effect on the motion of the system.

In order to arrive at meaningful conclusions about the dynamic motion of the towed system, it is important to understand the manner in which disturbances are propagated along the towline. Cristescu[7] indicates that, if a towline is not straight, each type of disturbance, transverse or longitudinal, influences the other. For instance, if the initial motion of the towing ship is directed along the towline, then one would expect a longitudinal disturbance to be propagated along the towline. However, because of variation of the towline shape, transverse waves will soon appear. This phenomenon is illustrated in Fig. 8. At point 1 the motion is tangent to the towline. As the disturbance proceeds along the towline to point 2, the towline motion develops a transverse component due to its curvature.

This coupling between the types of motion can be demonstrated with the use of the numerical solution of Sec. 3.2. If the motion at the upper end of the towline is purely longitudinal, then the motion at the lower end should be both transverse and longitudinal. Table 1 indicates that the motion at the lower end of the towline is a combination of transverse and longitudinal disturbances. Also, in Table 1, a purely transverse motion is imparted to the towline at its upper boundary, and the motion at its lower end is again made up of transverse and longitudinal modes. Therefore, one can say that longitudinal disturbances generate transverse motion and vice versa, when the towline is not straight.

In the aforementioned examples, the same disturbance amplitude, 10 ft, was applied at the upper end. From Table 1 it can be seen that both the transverse and longitudinal disturbances are diminished as they travel down the towline. This result is substantiated by the stability analysis of Sec. 5.0 of Ref. 9. Also, it should be noted that the transverse

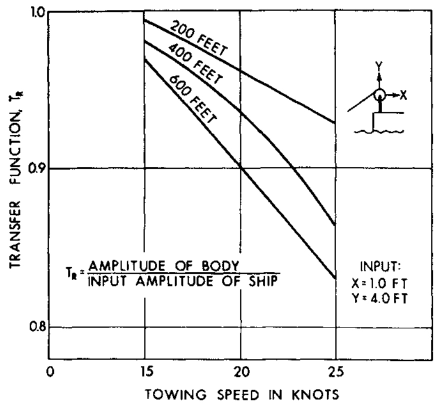

Fig. 9 Transfer function vs towing speed.

motion is damped to a much greater degree than the longitudinal motion.

When the numerical scheme of Sec. 3.2 was programmed on an IBM 7044 digital computer, the motion of the towing ship was assumed to be sinusoidal.‡ The Y displacement at the upper end was four times as large as the X displacement, and the displacements were considered to be in phase. Figure 9 represents the variation of the transfer function T_R, with towing speed and towline length. The transfer function is the ratio of the resultant amplitude of the body motion to the amplitude of the motion of the ship. It can be seen that the transfer function decreases as the towline length increases. This is caused by the fact that the disturbances have to travel a longer distance to the body, and, therefore, they are subjected to a greater amount of damping. Also, the transfer function decreases with increased towing speed. This results because the towline's pitch angle becomes smaller as the speed is increased. Therefore, a greater percentage of the input disturbance is transverse, and as indicated in Table 1, transverse disturbances are damped to a much larger extent than longitudinal disturbances.

‡ The actual ship motion could be put into the program. However, for convenience, the sinusoidal motion was used.

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